Lecture Note: KBM 603
Superconducting Qubits I
From Quantum Harmonic Oscillator to the Transmon

Contents

1 Review: Quantum Harmonic Oscillator (QHO)

Assumptions

We treat a single degree of freedom oscillator with Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$. Quantization is done by promoting $(x, p) \to (\hat{x}, \hat{p})$ with canonical commutation $[\hat{x}, \hat{p}] = i\hbar$.

1.1 Canonical quantization and goal

Classical Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1}$$

Quantize:

$$x \to \hat{x}, \qquad p \to \hat{p}, \qquad [\hat{x}, \hat{p}] = i\hbar. \tag{2}$$

Logic note

Our goal is to rewrite $\hat{H}$ in a form where eigenstates are obvious. For quadratic Hamiltonians, ladder operators achieve this.

1.2 Define ladder operators and verify algebra (all steps)

Define:

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2\hbar m\omega}}\,\hat{p}, \tag{3}$$

$$\hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2\hbar m\omega}}\,\hat{p}. \tag{4}$$

Step 1: compute $[\hat{a}, \hat{a}^\dagger]$. Let $\alpha = \sqrt{\frac{m\omega}{2\hbar}}$, $\beta = \frac{1}{\sqrt{2\hbar m\omega}}$, so $\hat{a} = \alpha\hat{x} + i\beta\hat{p}$ and $\hat{a}^\dagger = \alpha\hat{x} - i\beta\hat{p}$. Then

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= [\alpha\hat{x} + i\beta\hat{p},\ \alpha\hat{x} - i\beta\hat{p}] \\
&= \alpha^2[\hat{x}, \hat{x}] - i\alpha\beta[\hat{x}, \hat{p}] + i\alpha\beta[\hat{p}, \hat{x}] + (i\beta)(-i\beta)[\hat{p}, \hat{p}] \\
&= 0 - i\alpha\beta(i\hbar) + i\alpha\beta(-i\hbar) + 0 \\
&= \alpha\beta\hbar + \alpha\beta\hbar \\
&= 2\alpha\beta\hbar \\
&= 2\left(\sqrt{\frac{m\omega}{2\hbar}}\right)\left(\frac{1}{\sqrt{2\hbar m\omega}}\right)\hbar \\
&= 2\left(\frac{1}{2\hbar}\right)\hbar \\
&= 1.
\end{aligned} \tag{5}$$

$$\boxed{[\hat{a}, \hat{a}^\dagger] = 1.} \tag{6}$$

1.3 Invert to get $\hat{x}$ and $\hat{p}$ (explicit)

Add Eqs. (3) and (4):

$$\hat{a}+\hat{a}^\dagger = 2\sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \quad\Rightarrow\quad \hat{x}=\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger). \tag{7}$$

Subtract:

$$\hat{a}-\hat{a}^\dagger = 2\frac{i}{\sqrt{2\hbar m\omega}}\,\hat{p} \quad\Rightarrow\quad \hat{p}=\frac{\sqrt{2\hbar m\omega}}{2i}(\hat{a}-\hat{a}^\dagger)=i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger-\hat{a}). \tag{8}$$

Define zero-point fluctuation scales:

$$x_{\text{zpf}}\equiv\sqrt{\frac{\hbar}{2m\omega}}, \qquad p_{\text{zpf}}\equiv\sqrt{\frac{\hbar m\omega}{2}}. \tag{9}$$

Then

$$\boxed{\hat{x}=x_{\text{zpf}}(\hat{a}+\hat{a}^\dagger), \qquad \hat{p}=i\,p_{\text{zpf}}(\hat{a}^\dagger-\hat{a}).} \tag{10}$$

Logic note

These forms will be mirrored exactly in the LC resonator after substituting $(x,p)\leftrightarrow(\Phi,Q)$.

1.4 Derive $\hat{H}=\hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})$ step-by-step

Start with

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2. \tag{11}$$

Substitute $\hat{x}$ and $\hat{p}$ in terms of $\hat{a},\hat{a}^\dagger$.

Kinetic term.

$$\begin{aligned}
\frac{\hat{p}^2}{2m} &= \frac{1}{2m}\left(ip_{\text{zpf}}(\hat{a}^\dagger-\hat{a})\right)^2\\
&= \frac{1}{2m}\left(-p_{\text{zpf}}^2(\hat{a}^\dagger-\hat{a})^2\right)\\
&= -\frac{p_{\text{zpf}}^2}{2m}\left(\hat{a}^\dagger-\hat{a}\right)^2\\
&= -\frac{1}{2m}\left(\frac{\hbar m\omega}{2}\right)(\hat{a}^\dagger-\hat{a})^2\\
&= -\frac{\hbar\omega}{4}(\hat{a}^\dagger-\hat{a})^2.
\end{aligned} \tag{12}$$

Potential term.

$$\begin{aligned}\frac{1}{2}m\omega^2\hat{x}^2 &= \frac{1}{2}m\omega^2\left(x_{\text{zpf}}(\hat{a}+\hat{a}^\dagger)\right)^2\\ &= \frac{1}{2}m\omega^2 x_{\text{zpf}}^2(\hat{a}+\hat{a}^\dagger)^2\\ &= \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)(\hat{a}+\hat{a}^\dagger)^2\\ &= \frac{\hbar\omega}{4}(\hat{a}+\hat{a}^\dagger)^2. \qquad (13)\end{aligned}$$

Add them:

$$\hat{H} = \frac{\hbar\omega}{4}\Big[(\hat{a}+\hat{a}^\dagger)^2 - (\hat{a}^\dagger-\hat{a})^2\Big]. \qquad (14)$$

Expand both squares:

$$(\hat{a}+\hat{a}^\dagger)^2 = \hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2, \qquad (15)$$

$$(\hat{a}^\dagger-\hat{a})^2 = (\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2. \qquad (16)$$

Subtract:

$$\begin{aligned}(\hat{a}+\hat{a}^\dagger)^2 - (\hat{a}^\dagger-\hat{a})^2 &= \Big[\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + (\hat{a}^\dagger)^2\Big] - \Big[(\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2\Big]\\ &= 2(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}). \qquad (17)\end{aligned}$$

Plug into Eq. (14):

$$\hat{H} = \frac{\hbar\omega}{4}\cdot 2(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \frac{\hbar\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}). \qquad (18)$$

Now use $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + [\hat{a},\hat{a}^\dagger] = \hat{a}^\dagger\hat{a} + 1$:

$$\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = (\hat{a}^\dagger\hat{a} + 1) + \hat{a}^\dagger\hat{a} = 2\hat{a}^\dagger\hat{a} + 1. \qquad (19)$$

Therefore from Eq. (18):

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right).} \qquad (20)$$

Define number operator:

$$\boxed{\hat{n} \equiv \hat{a}^\dagger\hat{a}, \qquad \hat{n}\left|n\right\rangle = n\left|n\right\rangle.} \qquad (21)$$

Energy spectrum:

$$\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right).} \qquad (22)$$

Key points to remember

- QHO is any quadratic Hamiltonian in a conjugate pair.
- Ladder operators package the algebra and reveal the spectrum.
- Zero-point fluctuations: $x_{\text{zpf}} = \sqrt{\hbar/(2m\omega)}$.

2 Quantum LC Resonator (Harmonic Oscillator in Circuit Variables)

Assumptions

- **Lumped-element limit:** circuit dimensions $\ll \lambda$ so voltages/currents are spatially uniform per element.
- Lossless elements for Hamiltonian derivation (dissipation can be added later phenomenologically).
- Single-mode, single degree of freedom LC.

2.1 Classical energy and choice of canonical variables

Stored energies:

$$H_{LC} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}. \tag{23}$$

Define charge and flux:

$$Q(t) = \int^t I(t')\,dt', \qquad \Phi(t) = \int^t V(t')\,dt'. \tag{24}$$

Logic note

In circuits, Φ (time-integral of voltage) plays the role of coordinate, and Q (time-integral of current) plays the role of momentum.

Identify resonance frequency and impedance:

$$\omega_r = \frac{1}{\sqrt{LC}}, \qquad Z_r = \sqrt{\frac{L}{C}}. \tag{25}$$

2.2 Quantization: $[\hat{\Phi}, \hat{Q}] = i\hbar$ and ladder operators

Promote: $\Phi \to \hat{\Phi}$, $Q \to \hat{Q}$, impose canonical commutation:

$$\boxed{[\hat{\Phi}, \hat{Q}] = i\hbar.} \tag{26}$$

We now choose ladder-operator parametrization exactly mirroring the QHO:

$$\hat{\Phi} = \Phi_{\text{zpf}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{Q} = i\,Q_{\text{zpf}}(\hat{a}^\dagger - \hat{a}). \tag{27}$$

Step 1: determine Φ_{zpf} and Q_{zpf} from the commutator. Compute:

$$\begin{aligned}
[\hat{\Phi}, \hat{Q}] &= [\Phi_{\text{zpf}}(\hat{a} + \hat{a}^\dagger),\ iQ_{\text{zpf}}(\hat{a}^\dagger - \hat{a})] \\
&= i\Phi_{\text{zpf}}Q_{\text{zpf}}\,[(\hat{a} + \hat{a}^\dagger), (\hat{a}^\dagger - \hat{a})] \\
&= i\Phi_{\text{zpf}}Q_{\text{zpf}}\Big([\hat{a}, \hat{a}^\dagger] - [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}^\dagger] - [\hat{a}^\dagger, \hat{a}]\Big) \\
&= i\Phi_{\text{zpf}}Q_{\text{zpf}}\Big(1 - 0 - 0 - (-1)\Big) \\
&= i\Phi_{\text{zpf}}Q_{\text{zpf}} \cdot 2 \\
&= 2i\Phi_{\text{zpf}}Q_{\text{zpf}}.
\end{aligned} \tag{28}$$

To satisfy Eq. (26), we require:

$$2\Phi_{\text{zpf}}Q_{\text{zpf}} = \hbar \quad \Rightarrow \quad \boxed{\Phi_{\text{zpf}}Q_{\text{zpf}} = \frac{\hbar}{2}.} \tag{29}$$

Step 2: match Hamiltonian coefficients to fix each zpf separately. Insert Eq. (27) into the LC Hamiltonian (23):

$$\begin{aligned}
\hat{H}_{LC} &= \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} \\
&= \frac{1}{2C}\left(iQ_{\text{zpf}}(\hat{a}^\dagger - \hat{a})\right)^2 + \frac{1}{2L}\left(\Phi_{\text{zpf}}(\hat{a} + \hat{a}^\dagger)\right)^2 \\
&= -\frac{Q_{\text{zpf}}^2}{2C}(\hat{a}^\dagger - \hat{a})^2 + \frac{\Phi_{\text{zpf}}^2}{2L}(\hat{a} + \hat{a}^\dagger)^2.
\end{aligned} \tag{30}$$

We want $\hat{H}_{LC}$ to reduce to $\hbar\omega_r(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. As in the QHO derivation, this happens if the prefactors are equal:

$$\frac{Q_{\text{zpf}}^2}{C} = \frac{\Phi_{\text{zpf}}^2}{L}. \tag{31}$$

Solve Eq. (31):

$$\frac{Q_{\text{zpf}}}{\Phi_{\text{zpf}}} = \sqrt{\frac{C}{L}} = \frac{1}{Z_r}. \tag{32}$$

Combine Eq. (29) and (32):

$$Q_{\text{zpf}} = \frac{\hbar}{2\Phi_{\text{zpf}}}, \tag{33}$$

$$\frac{\hbar}{2\Phi_{\text{zpf}}^2} = \frac{1}{Z_r} \quad \Rightarrow \quad \Phi_{\text{zpf}}^2 = \frac{\hbar Z_r}{2}. \tag{34}$$

Thus

$$\boxed{\Phi_{\text{zpf}} = \sqrt{\frac{\hbar Z_r}{2}}, \qquad Q_{\text{zpf}} = \sqrt{\frac{\hbar}{2Z_r}}.} \tag{35}$$

Step 3: final Hamiltonian form (show the same algebra as QHO). With matched prefactors, Eq. (30) becomes

$$\hat{H}_{LC} = \frac{\hbar\omega_r}{4}\Big[(\hat{a} + \hat{a}^\dagger)^2 - (\hat{a}^\dagger - \hat{a})^2\Big], \tag{36}$$

and repeating the QHO difference-of-squares algebra yields:

$$\boxed{\hat{H}_{LC} = \hbar\omega_r\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right).} \tag{37}$$

Key points to remember

- LC resonator is exactly a quantum harmonic oscillator with (Φ, Q) as the conjugate pair.
- Vacuum fluctuations: $\Phi_{\text{zpf}} = \sqrt{\hbar Z_r/2}$ and $Q_{\text{zpf}} = \sqrt{\hbar/(2Z_r)}$.
- Equal level spacing $\Rightarrow$ LC alone is *not* a qubit.

3 Why We Need Nonlinearity

For a harmonic oscillator,

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad \Rightarrow \quad \omega_{01} = \omega_{12} = \omega_{23} = \cdots = \omega. \tag{38}$$

If we drive at ω_{01}, nothing prevents excitation of higher transitions (leakage).

Logic note

A qubit needs a *spectrally isolated* transition. That means **anharmonicity**: $\omega_{12} \neq \omega_{01}$.

4 Josephson Junction as a Nonlinear Inductor (Derive the Cosine Energy)

4.1 Current-phase and flux-phase relations

Josephson current relation:

$$I(t) = I_c \sin\varphi(t). \tag{39}$$

Flux quantum:

$$\Phi_0 = \frac{h}{2e}. \tag{40}$$

Phase–flux relation across the junction:

$$\varphi(t) = \frac{2\pi}{\Phi_0}\Phi(t). \tag{41}$$

4.2 Energy stored in the junction from power (all steps)

Power delivered to the element:

$$P(t) = V(t)I(t). \tag{42}$$

Energy (work) stored:

$$E = \int P(t)\,dt = \int V(t)I(t)\,dt. \tag{43}$$

Using flux definition $V(t) = \dot{\Phi}(t)$:

$$E = \int \dot{\Phi}(t)\,I(t)\,dt. \tag{44}$$

Substitute $I(t) = I_c \sin\varphi(t)$ and $\varphi = (2\pi/\Phi_0)\Phi$:

$$E = I_c \int \dot{\Phi}(t)\,\sin\!\left(\frac{2\pi}{\Phi_0}\Phi(t)\right)\,dt. \tag{45}$$

Now perform substitution:

$$u(t) \equiv \frac{2\pi}{\Phi_0}\Phi(t) \quad\Rightarrow\quad \dot{u}(t) = \frac{2\pi}{\Phi_0}\dot{\Phi}(t) \quad\Rightarrow\quad \dot{\Phi}(t) = \frac{\Phi_0}{2\pi}\dot{u}(t).$$

Then Eq. (45) becomes

$$\begin{aligned} E &= I_c \int \frac{\Phi_0}{2\pi}\dot{u}(t)\,\sin(u(t))\,dt \\ &= \frac{I_c\Phi_0}{2\pi}\int \sin(u)\,du \\ &= -\frac{I_c\Phi_0}{2\pi}\cos u + \text{constant}. \end{aligned} \tag{46}$$

Define the Josephson energy

$$\boxed{E_J \equiv \frac{I_c\Phi_0}{2\pi}}, \tag{47}$$

and use $u = \varphi$ to write the stored energy (potential energy):

$$\boxed{U_J(\varphi) = -E_J\cos\varphi} \qquad \text{(up to an additive constant)}. \tag{48}$$

Key points to remember

- Linear inductor: $U_L = \Phi^2/(2L)$ (quadratic) ⇒ harmonic spectrum.
- Josephson junction: $U_J(\varphi) = -E_J\cos\varphi$ (cosine) ⇒ anharmonic spectrum.

5 Transmon Qubit: Replace L with Josephson Junction + Large Shunt Capacitance

Assumptions

- Single junction shunted by a capacitance C_Σ (includes junction capacitance + explicit shunt).
- We ignore dissipation in the Hamiltonian model.
- Offset charge n_g is included to discuss charge-noise sensitivity.

5.1 Classical Hamiltonian in (Φ, Q) (explicit)

Total energy = capacitor energy + Josephson energy:

$$H = \frac{(Q - Q_g)^2}{2C_\Sigma} - E_J \cos\left(\frac{2\pi}{\Phi_0}\Phi\right). \tag{49}$$

Here Q_g is the gate-induced (offset) charge.

5.2 Quantize and change variables to (φ, n)

Promote to operators and impose:

$$[\hat{\Phi}, \hat{Q}] = i\hbar. \tag{50}$$

Define dimensionless phase operator and Cooper-pair number operator:

$$\boxed{\hat{\varphi} \equiv \frac{2\pi}{\Phi_0}\hat{\Phi}, \qquad \hat{n} \equiv \frac{\hat{Q}}{2e}.} \tag{51}$$

Also define dimensionless offset charge:

$$\boxed{n_g \equiv \frac{Q_g}{2e}.} \tag{52}$$

Step: derive the commutator $[\hat{\varphi}, \hat{n}] = i$. Using Eq. (51):

$$\begin{aligned} [\hat{\varphi}, \hat{n}] &= \left[\frac{2\pi}{\Phi_0}\hat{\Phi},\ \frac{1}{2e}\hat{Q}\right] = \frac{2\pi}{\Phi_0}\frac{1}{2e}[\hat{\Phi}, \hat{Q}] \\ &= \frac{2\pi}{\Phi_0}\frac{1}{2e}(i\hbar) = i\hbar\frac{2\pi}{(h/2e)}\frac{1}{2e} \\ &= i\hbar\frac{2\pi}{h} \cdot 2e \cdot \frac{1}{2e} = i\hbar\frac{2\pi}{h} = i\hbar\frac{1}{\hbar} = i. \end{aligned} \tag{53}$$

So

$$\boxed{[\hat{\varphi}, \hat{n}] = i.} \tag{54}$$

5.3 Identify charging energy and write the standard transmon Hamiltonian

Define charging energy:

$$\boxed{E_C \equiv \frac{e^2}{2C_\Sigma}.} \tag{55}$$

Rewrite the capacitive term:

$$\frac{(\hat{Q} - Q_g)^2}{2C_\Sigma} = \frac{(2e(\hat{n} - n_g))^2}{2C_\Sigma} = \frac{4e^2}{2C_\Sigma}(\hat{n} - n_g)^2 = 4E_C(\hat{n} - n_g)^2. \tag{56}$$

Also rewrite the cosine using $\hat{\varphi} = (2\pi/\Phi_0)\hat{\Phi}$. Therefore the quantum Hamiltonian is

$$\boxed{\hat{H}_T = 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}.} \tag{57}$$

Logic note

This is the key result: **kinetic energy** is the charging term (in n), and **potential energy** is the Josephson cosine (in φ). The transmon is a "quantum particle" moving in a cosine potential.

6 Why E_J/E_C Matters: Noise Sensitivity vs Anharmonicity

6.1 Qualitative regimes

- **Charge regime** ($E_J/E_C \ll 1$): charging dominates, eigenstates are close to charge states $|n\rangle$, strong dependence on $n_g \Rightarrow$ charge noise sensitive.
- **Transmon regime** ($E_J/E_C \gg 1$): cosine dominates, phase localizes near a minimum, dependence on n_g becomes exponentially small (charge dispersion suppressed), while anharmonicity decreases slowly.

Key points to remember

- Increasing E_J/E_C strongly suppresses charge-noise sensitivity (good coherence).
- Too large E_J/E_C reduces anharmonicity (harder spectral selectivity), but only mildly compared to noise suppression.

7 Approximate Transmon Spectrum (Detailed Steps)

Assumptions

We work in the transmon regime $E_J/E_C \gg 1$ where low-energy states are localized in a single cosine well and phase fluctuations are small: $\Delta\varphi \ll 1$. We set $n_g \simeq 0$ when focusing on local well physics (good approximation in transmon regime due to weak charge dispersion).

7.1 Taylor expand the cosine and separate harmonic + perturbation

Start from Eq. (57) (with $n_g \approx 0$):

$$\hat{H} \approx 4E_C\hat{n}^2 - E_J \cos\hat{\varphi}. \tag{58}$$

Expand around a minimum at $\varphi = 0$:

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots \tag{59}$$

So

$$\begin{aligned}-E_J\cos\varphi &= -E_J\left(1-\frac{\varphi^2}{2}+\frac{\varphi^4}{24}-\cdots\right)\\ &= -E_J+\frac{E_J}{2}\varphi^2-\frac{E_J}{24}\varphi^4+\cdots\end{aligned} \tag{60}$$

Hence

$$\hat{H}\approx\underbrace{4E_C\hat{n}^2+\frac{E_J}{2}\hat{\varphi}^2}_{\text{harmonic part }\hat{H}_0}\quad\underbrace{-\frac{E_J}{24}\hat{\varphi}^4}_{\text{anharmonic perturbation }\hat{V}}\quad\underbrace{-E_J}_{\text{constant shift}}. \tag{61}$$

We can drop the constant $-E_J$ since it shifts all energies equally.

Logic note

At this stage the transmon looks like a harmonic oscillator in (φ, n) plus a small quartic correction. That quartic term is what makes it a qubit.

7.2 Quantize the harmonic part by defining $b, b^\dagger$

We want to map $\hat{H}_0 = 4E_C\hat{n}^2 + \frac{E_J}{2}\hat{\varphi}^2$ to $\hbar\omega_p\left(\hat{b}^\dagger\hat{b}+\frac{1}{2}\right)$ for some plasma frequency ω_p.

Introduce ladder operators $\hat{b}, \hat{b}^\dagger$ via

$$\boxed{\hat{\varphi}=\varphi_{\text{zpf}}(\hat{b}+\hat{b}^\dagger),\qquad \hat{n}=in_{\text{zpf}}(\hat{b}^\dagger-\hat{b})} \tag{62}$$

with $[\hat{b},\hat{b}^\dagger]=1$.

Step 1: enforce $[\hat{\varphi},\hat{n}]=i$ to relate φ_{zpf} and n_{zpf}. Compute:

$$\begin{aligned}[\hat{\varphi},\hat{n}] &= [\varphi_{\text{zpf}}(\hat{b}+\hat{b}^\dagger),\ in_{\text{zpf}}(\hat{b}^\dagger-\hat{b})]\\ &= i\varphi_{\text{zpf}}n_{\text{zpf}}[(\hat{b}+\hat{b}^\dagger),(\hat{b}^\dagger-\hat{b})]\\ &= i\varphi_{\text{zpf}}n_{\text{zpf}}\cdot 2\\ &= 2i\varphi_{\text{zpf}}n_{\text{zpf}}.\end{aligned} \tag{63}$$

Thus

$$\boxed{\varphi_{\text{zpf}}n_{\text{zpf}}=\frac{1}{2}.} \tag{64}$$

Step 2: match coefficients to make $\hat{H}_0$ exactly harmonic. Compute $\hat{n}^2$ and $\hat{\varphi}^2$ contributions:

$$4E_C\hat{n}^2 = 4E_C\left(in_{\text{zpf}}(\hat{b}^\dagger-\hat{b})\right)^2 = -4E_Cn_{\text{zpf}}^2(\hat{b}^\dagger-\hat{b})^2, \tag{65}$$

$$\frac{E_J}{2}\hat{\varphi}^2=\frac{E_J}{2}\left(\varphi_{\text{zpf}}(\hat{b}+\hat{b}^\dagger)\right)^2=\frac{E_J}{2}\varphi_{\text{zpf}}^2(\hat{b}+\hat{b}^\dagger)^2. \tag{66}$$

As in QHO, we want equal prefactors (so the difference-of-squares collapses to number operator):

$$4E_Cn_{\text{zpf}}^2=\frac{E_J}{2}\varphi_{\text{zpf}}^2. \tag{67}$$

Use Eq. (64) ($n_{\text{zpf}} = 1/(2\varphi_{\text{zpf}})$) in Eq. (67):

$$\begin{aligned}
4E_C\left(\frac{1}{2\varphi_{\text{zpf}}}\right)^2 &= \frac{E_J}{2}\varphi_{\text{zpf}}^2 \\
4E_C\frac{1}{4\varphi_{\text{zpf}}^2} &= \frac{E_J}{2}\varphi_{\text{zpf}}^2 \\
\frac{E_C}{\varphi_{\text{zpf}}^2} &= \frac{E_J}{2}\varphi_{\text{zpf}}^2 \\
E_C &= \frac{E_J}{2}\varphi_{\text{zpf}}^4 \\
\varphi_{\text{zpf}}^4 &= \frac{2E_C}{E_J}. \qquad (68)
\end{aligned}$$

Therefore

$$\boxed{\varphi_{\text{zpf}} = \left(\frac{2E_C}{E_J}\right)^{1/4}, \qquad n_{\text{zpf}} = \frac{1}{2}\left(\frac{E_J}{2E_C}\right)^{1/4}.} \tag{69}$$

Step 3: obtain the plasma frequency ω_p. With matched prefactors, the same algebra as QHO yields

$$\hat{H}_0 = \hbar\omega_p\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right), \tag{70}$$

where the frequency comes from the product of the quadratic coefficients:

$$\boxed{\hbar\omega_p = \sqrt{8E_CE_J}.} \tag{71}$$

7.3 Include the quartic term and extract anharmonicity (explicit)

From Eq. (61), the perturbation is

$$\hat{V} = -\frac{E_J}{24}\hat{\varphi}^4. \tag{72}$$

Using $\hat{\varphi} = \varphi_{\text{zpf}}(\hat{b} + \hat{b}^\dagger)$:

$$\hat{V} = -\frac{E_J}{24}\varphi_{\text{zpf}}^4(\hat{b} + \hat{b}^\dagger)^4. \tag{73}$$

But $\varphi_{\text{zpf}}^4 = 2E_C/E_J$ from Eq. (69), so

$$\boxed{\hat{V} = -\frac{E_C}{12}(\hat{b} + \hat{b}^\dagger)^4.} \tag{74}$$

Rotating-wave (number-conserving) approximation: keep only terms that conserve excitation number. Expand $(\hat{b} + \hat{b}^\dagger)^4$ and retain only number-conserving contributions. The key surviving operator is $\hat{b}^{\dagger 2}\hat{b}^2$, plus terms that shift the frequency.

A standard result (from normal ordering / RWA) gives the effective Kerr form:

$$\boxed{\hat{H} \approx \hbar\omega_p\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) - \frac{E_C}{2}\hat{b}^{\dagger 2}\hat{b}^2 - \frac{E_C}{2}\hat{b}^\dagger\hat{b} \ + \ \text{const.}} \tag{75}$$

Grouping the linear-in-number term into the frequency yields the transmon transition frequency

$$\boxed{\hbar\omega_{01} \approx \sqrt{8E_C E_J} - E_C.} \tag{76}$$

The quartic Kerr term shifts higher levels by an amount proportional to $m(m-1)$, leading to

$$E_m \approx -E_J + \hbar\omega_{01}\, m - \frac{E_C}{2} m(m-1), \tag{77}$$

and therefore the anharmonicity

$$\boxed{\alpha \equiv \omega_{12} - \omega_{01} \approx -\frac{E_C}{\hbar}.} \tag{78}$$

Logic note

The practical meaning of $\alpha < 0$: the 1→2 transition is *lower* in frequency than 0→1, so a resonant π-pulse at ω_{01} is less likely to excite $|2\rangle$ if $|\alpha|$ is large enough compared to the drive bandwidth.

Key points to remember

- Harmonic approximation gives plasma frequency: $\hbar\omega_p = \sqrt{8E_C E_J}$.
- First correction shifts the qubit frequency: $\hbar\omega_{01} \approx \sqrt{8E_C E_J} - E_C$.
- Anharmonicity: $\alpha \approx -E_C/\hbar$ is the "spectral selectivity" knob.
- Increasing E_J/E_C reduces charge sensitivity strongly while keeping α usable.

8 Summary (One-Page Lecture Recall)

Final recall

- **QHO:** $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ with $[\hat{a}, \hat{a}^\dagger] = 1$.
- **LC resonator:** $\hat{H}_{LC} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}$, $[\hat{\Phi}, \hat{Q}] = i\hbar$, and it is a QHO with $\omega_r = 1/\sqrt{LC}$.
- **Josephson junction:** $U_J(\varphi) = -E_J \cos\varphi$, $E_J = \Phi_0 I_c/(2\pi)$.
- **Transmon:** $\hat{H}_T = 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}$, $[\hat{\varphi}, \hat{n}] = i$.
- **In transmon regime:** $\hbar\omega_{01} \approx \sqrt{8E_C E_J} - E_C$, and $\alpha \approx -E_C/\hbar$.

9 Circuit QED: Coupling a Transmon to a Microwave Resonator

Assumptions

- We consider a single resonator mode (frequency ω_r) coupled to a single transmon mode.
- Coupling is capacitive and sufficiently weak that a linear (dipole) interaction is valid.
- We first keep the transmon as a weakly anharmonic oscillator, then reduce it to a two-level system (TLS) when appropriate.

9.1 Physical picture and variables

A superconducting qubit (transmon) can be coupled to a quantized microwave resonator. The resonator is a quantum LC oscillator:

$$\hat{H}_r = \hbar\omega_r \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \tag{79}$$

with ladder operators $\hat{a}, \hat{a}^\dagger$.

The transmon Hamiltonian (from previous sections) is

$$\hat{H}_T = 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}. \tag{80}$$

In the transmon regime, we can approximate it as a weakly anharmonic oscillator (Kerr oscillator):

$$\hat{H}_T \approx \hbar\omega_q \hat{b}^\dagger \hat{b} - \frac{E_C}{2} \hat{b}^{\dagger 2} \hat{b}^2 \quad (+\text{ constants}), \tag{81}$$

where $\hat{b}$ is the transmon lowering operator and $\omega_q \approx (\sqrt{8E_C E_J} - E_C)/\hbar$.

Logic note

At this stage, the system is "two oscillators" (resonator + transmon mode), but the transmon has a weak nonlinearity. That nonlinearity is what enables qubit behavior and dispersive readout.

9.2 How capacitive coupling generates an interaction term

The most common cQED implementation is **capacitive coupling**. A coupling capacitor C_g connects the resonator voltage node to the transmon island. The interaction energy of a capacitor is (classically):

$$E_C^{(\text{cap})} = \frac{(Q)^2}{2C}. \tag{82}$$

When two nodes are connected by a coupling capacitor, the node charges become mixed. After writing the full circuit Lagrangian and performing a Legendre transform, the Hamiltonian contains a cross term of the form

$$H_{\text{int}} \propto Q_r Q_q, \tag{83}$$

i.e. resonator charge × transmon charge.

Quantize:

$$Q_r \to \hat{Q}_r, \qquad Q_q \to \hat{Q}_q.$$

Use the resonator operator form

$$\hat{Q}_r = iQ_{r,\text{zpf}}(\hat{a}^\dagger - \hat{a}), \tag{84}$$

and the transmon charge operator

$$\hat{Q}_q = 2e\,\hat{n}. \tag{85}$$

Thus the generic interaction becomes

$$\hat{H}_\text{int} = \hbar g\,(\hat{a} + \hat{a}^\dagger)\,\hat{n}, \tag{86}$$

where g (with units of frequency) is a coupling constant set by circuit parameters (capacitances, impedances, participation ratios).

Logic note

The resonator provides a quantum electric field (voltage fluctuations). The transmon couples to that field through its electric dipole moment, which is proportional to $\hat{n}$.

9.3 Jaynes–Cummings limit (two-level approximation)

To obtain the standard cQED Hamiltonian, we reduce the transmon to its lowest two levels:

$$|g\rangle\,, |e\rangle\,.$$

Within this subspace, define Pauli operators:

$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|\,, \qquad \hat{\sigma}_+ = |e\rangle\langle g|\,, \qquad \hat{\sigma}_- = |g\rangle\langle e|\,.$$

The charge operator $\hat{n}$ has matrix elements between $|g\rangle$ and $|e\rangle$:

$$\hat{n} \;\to\; n_{ge}\,(\hat{\sigma}_- + \hat{\sigma}_+) \qquad \text{(two-level truncation)}, \tag{87}$$

where $n_{ge} = \langle g|\hat{n}|e\rangle$ is a device-dependent dipole matrix element.

Insert Eq. (87) into Eq. (86):

$$\hat{H}_\text{int} = \hbar g\,(\hat{a} + \hat{a}^\dagger)\,n_{ge}(\hat{\sigma}_- + \hat{\sigma}_+). \tag{88}$$

Define the effective coupling

$$\boxed{g_\text{eff} \equiv g\,n_{ge}.} \tag{89}$$

Then

$$\hat{H}_\text{int} = \hbar g_\text{eff}(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_- + \hat{\sigma}_+). \tag{90}$$

Rotating-wave approximation (RWA). In the interaction picture, terms like $\hat{a}\hat{\sigma}_-$ and $\hat{a}^\dagger\hat{\sigma}_+$ oscillate near $e^{-i(\omega_r+\omega_q)t}$ and average out when $g_{\text{eff}} \ll \omega_r, \omega_q$. Keep only energy-conserving terms $\hat{a}\hat{\sigma}_+$ and $\hat{a}^\dagger\hat{\sigma}_-$:

$$\boxed{\hat{H}_{\text{int}}^{(\text{RWA})} = \hbar g_{\text{eff}} \left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right).} \tag{91}$$

The transmon (TLS) Hamiltonian is

$$\hat{H}_q = \frac{\hbar\omega_q}{2}\hat{\sigma}_z. \tag{92}$$

Thus the **Jaynes–Cummings Hamiltonian** is

$$\boxed{\hat{H}_{\text{JC}} = \hbar\omega_r\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_q}{2}\hat{\sigma}_z + \hbar g_{\text{eff}}\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right).} \tag{93}$$

9.4 Dispersive regime and effective Hamiltonian (why cQED enables readout)

Define detuning:

$$\Delta \equiv \omega_q - \omega_r. \tag{94}$$

In the **dispersive regime**,

$$\boxed{|\Delta| \gg g_{\text{eff}},} \tag{95}$$

real energy exchange is suppressed, but virtual exchange shifts frequencies. To second order in g_{eff}/Δ, the effective Hamiltonian becomes

$$\boxed{\hat{H}_{\text{disp}} \approx \hbar\left(\omega_r + \chi\,\hat{\sigma}_z\right)\hat{a}^\dagger\hat{a} + \frac{\hbar}{2}\left(\omega_q + \chi\right)\hat{\sigma}_z,} \tag{96}$$

where χ is the **dispersive shift**.

Logic note

The key measurement idea: the resonator frequency depends on the qubit state. If the qubit is in $|g\rangle$ the resonator is at $\omega_r - \chi$; if in $|e\rangle$ it is at $\omega_r + \chi$. Measuring the resonator response gives the qubit state.

Including transmon anharmonicity (important correction). Because the transmon is not an ideal two-level system, χ differs from the simple TLS formula. A widely used approximation for a weakly anharmonic transmon is

$$\boxed{\chi \approx -\frac{g_{\text{eff}}^2}{\Delta}\frac{\alpha}{\Delta + \alpha},} \tag{97}$$

where $\alpha \approx -E_C/\hbar$ is the transmon anharmonicity (negative).

Key points to remember

- cQED couples a qubit to a resonator: $\hat{H} = \hat{H}_r + \hat{H}_q + \hat{H}_{\text{int}}$.
- Capacitive coupling produces an interaction $\propto (\hat{a} + \hat{a}^\dagger)\hat{n}$.
- Two-level + RWA gives Jaynes–Cummings: $\hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)$.
- In dispersive regime ($|\Delta| \gg g$): resonator frequency shifts by $\pm\chi$ depending on qubit state $\Rightarrow$ readout.
- Transmon anharmonicity modifies χ and is essential for quantitative cQED modeling.